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Functional representations of integrable hierarchies

Aristophanes Dimakis[1] **and Folkert Müller-Hoissen**[2]

[1] Department of Financial and Management Engineering, University of the Aegean, 31 Fostini Str., GR-82100 Chios, Greece
[2] Max-Planck-Institute for Dynamics and Self-Organization, Bunsenstrasse 10, D-37073 Göttingen, Germany

E-mail: dimakis@aegean.gr and fmuelle@gwdg.de

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Abstract
We consider a general framework for integrable hierarchies in Lax form and derive certain universal equations from which 'functional representations' of particular hierarchies (such as KP, discrete KP, mKP, AKNS), i.e. formulations in terms of functional equations, are systematically and quite easily obtained. The formalism genuinely applies to hierarchies where the dependent variables live in a noncommutative (typically matrix) algebra. The obtained functional representations can be understood as 'noncommutative' analogues of 'Fay identities' for the KP hierarchy.

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1. Introduction

In the framework of Gelfand–Dickey-type hierarchies [1] (see also section 2.4), the commutativity of flows, which is the hierarchy property, is an almost trivial consequence. On the other hand, one is dealing with a rather implicit form of flow equations and it is quite difficult to extract them in more explicit form. In the case where the dependent variables take their values in the (commutative) algebra of functions (of the infinite set of evolution times), expressions of the hierarchy in terms of (Hirota–Sato) τ-functions can typically be achieved. For example, the famous KP hierarchy in Gelfand–Dickey-form $L_{t_n} = [(L^n)_+, L]$ (see section 2.4 for notational details) is equivalent to a 'Fay identity' (see [1–9], in particular). Such a representation of the hierarchy in the form of functional equations expresses the complete set of hierarchy equations directly in terms of the relevant dependent variables as a system of equations which depend on auxiliary parameters (see also [10–19] for related work).

In a recent publication [20], we were led to a formula which may be regarded as a counterpart of the 'differential Fay identity' in the case of the KP hierarchy with variables in a *non*commutative algebra, e.g., an algebra of matrices of functions. In this work, we consider the correspondence between such 'noncommutative' (and in particular Gelfand–Dickey-type) hierarchies and equations which may be regarded as 'noncommutative Fay identities'. The main results can actually be proved in a surprisingly general setting. A more specialized framework then allows us to apply the general results simultaneously in particular to the KP, discrete KP, q-KP, AKNS and other hierarchies.

In section 2, we start with a quite general framework for integrable hierarchies. In subsection 2.4, we specialize it to Gelfand–Dickey-type hierarchies and prove a central result of this work. Section 3 then concentrates on a more concrete class of examples. A modified KP hierarchy is treated in section 4. Finally, section 5 contains some concluding remarks.

2. A general framework for hierarchies

2.1. Preliminaries

Let $\mathbf{t} = (t_1, t_2, t_3, \ldots)$ be a set of independent (commuting) variables. We introduce

$$\chi(\lambda) := \exp\left(\sum_{n\geqslant 1} \frac{\lambda^n}{n} \partial_{t_n}\right) =: \sum_{n\geqslant 0} \lambda^n \chi_n, \qquad \chi(\lambda)^{-1} =: \sum_{n\geqslant 0} \lambda^n \hat{\chi}_n \tag{2.1}$$

as formal power series in some auxiliary parameter λ. Then[3]

$$\chi_n = p_n(\tilde{\partial}), \qquad \hat{\chi}_n = p_n(-\tilde{\partial}), \qquad n = 0, 1, 2, \ldots \tag{2.2}$$

where $p_n, n = 0, 1, 2, \ldots,$ are the elementary Schur polynomials (see [21, 22], for example), and

$$\tilde{\partial} := \left(\partial_{t_1}, \partial_{t_2}/2, \partial_{t_3}/3, \ldots\right). \tag{2.3}$$

If F depends on $\mathbf{t}$, then

$$\chi(\lambda)(F) = F(\mathbf{t} + [\lambda]) =: F_{[\lambda]}(\mathbf{t}), \qquad \chi(\lambda)^{-1}(F) = F(\mathbf{t} - [\lambda]) =: F_{-[\lambda]}(\mathbf{t}), \tag{2.4}$$

where

$$[\lambda] := (\lambda, \lambda^2/2, \lambda^3/3, \ldots). \tag{2.5}$$

Using

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\chi(\lambda) = \partial(\lambda)\chi(\lambda) \tag{2.6}$$

with the derivation

$$\partial(\lambda) := \sum_{n\geqslant 1} \lambda^{n-1}\partial_{t_n}, \tag{2.7}$$

we find

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} F_{[\lambda]} = \partial(\lambda)(F_{[\lambda]}), \qquad \frac{\mathrm{d}}{\mathrm{d}\lambda} F_{-[\lambda]} = -\partial(\lambda)(F_{-[\lambda]}). \tag{2.8}$$

Furthermore, from (2.6) we obtain

$$n\chi_n = \sum_{k=1}^{n} \partial_{t_k}\chi_{n-k}, \qquad n\hat{\chi}_n = -\sum_{k=1}^{n} \partial_{t_k}\hat{\chi}_{n-k}, \qquad n = 1, 2, \ldots. \tag{2.9}$$

Since, as an exponential of a derivation, $\chi(\lambda)$ is an automorphism, we have

$$\chi(\lambda)(FG) = \chi(\lambda)(F)\chi(\lambda)(G) \tag{2.10}$$

[3] Note that $\chi_0 = \mathrm{id} = \hat{\chi}_0$ and $\chi_1 = \partial_{t_1} = -\hat{\chi}_1$.

on elements F, G of an algebra, the elements of which depend on $\mathbf{t}$. As a consequence, χ_n and $\hat{\chi}_n$, $n = 0, 1, 2, \ldots$, are Hasse–Schmidt derivations [23, 24], i.e. they satisfy the generalized Leibniz rules

$$\chi_n(FG) = \sum_{k=0}^{n} \chi_k(F)\chi_{n-k}(G), \qquad \hat{\chi}_n(FG) = \sum_{k=0}^{n} \hat{\chi}_k(F)\hat{\chi}_{n-k}(G). \tag{2.11}$$

2.2. Linear systems and their integrability conditions

The integrability conditions of the linear system

$$\partial_{t_n}(W) = L_n W, \qquad n = 1, 2, \ldots \tag{2.12}$$

with invertible[4] W are

$$\partial_{t_m}(L_n) - \partial_{t_n}(L_m) = [L_m, L_n] \tag{2.13}$$

('zero curvature' or 'Zakharov–Shabat' conditions). Here, L_n and W are elements of a unital algebra $\mathfrak{R}$. Let us rewrite the linear system in the following way[5]:

$$\hat{\chi}_n(W) = E_n W, \qquad n = 0, 1, 2, \ldots \tag{2.14}$$

with $E_n \in \mathfrak{R}$. Then $E_0 = 1$ (where 1 stands here for the unit in $\mathfrak{R}$), $E_1 = -L_1$, $E_2 = (1/2)\big(-L_2 + \partial_{t_1}(L_1) + L_1 L_1\big)$, and so forth, so that E_n can be expressed in terms of L_k, $k \leqslant n$, and their derivatives. Introducing

$$E(\lambda) := \sum_{n \geqslant 0} \lambda^n E_n, \tag{2.15}$$

the linear system takes the form

$$W_{-[\lambda]} = E(\lambda) W. \tag{2.16}$$

As a consequence, we have

$$W_{-[\lambda_1]-[\lambda_2]} = E(\lambda_2)_{-[\lambda_1]} W_{-[\lambda_1]} = E(\lambda_2)_{-[\lambda_1]} E(\lambda_1) W \tag{2.17}$$

which requires the last expression to be symmetric in λ_1, λ_2. Hence, the integrability conditions of the linear system translate to

$$E(\lambda_2)_{-[\lambda_1]} E(\lambda_1) = E(\lambda_1)_{-[\lambda_2]} E(\lambda_2). \tag{2.18}$$

This formula is of central importance in this work. Expanding in λ_1, λ_2, we obtain the following expression of the zero curvature conditions:

$$\sum_{k=0}^{m} \hat{\chi}_k(E_n) E_{m-k} = \sum_{k=0}^{n} \hat{\chi}_k(E_m) E_{n-k} \qquad m, n = 0, 1, 2, \ldots. \tag{2.19}$$

By the use of (2.8) and (2.12), we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(E(\lambda)W) = -\partial(\lambda)(W_{-[\lambda]}) = -\partial(\lambda)(E(\lambda)W) = -\partial(\lambda)(E(\lambda))W - E(\lambda)L(\lambda)W, \tag{2.20}$$

where

$$L(\lambda) := \sum_{n \geqslant 1} \lambda^{n-1} L_n. \tag{2.21}$$

[4] W should be regarded as a 'fundamental matrix solution' of the linear system.

[5] We may express the linear system alternatively and equivalently in the form $\chi_n(W) = H_n W$ with $H_n \in \mathfrak{R}$. Our choice turns out to be more convenient, however. See the remark in section 3.2.

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} E(\lambda) = -\partial(\lambda)(E(\lambda)) - E(\lambda)L(\lambda). \tag{2.22}$$

Expansion in powers of λ leads to

$$nE_n = -\sum_{k=1}^{n} \big(\partial_{t_k}(E_{n-k}) + E_{n-k}L_k\big), \qquad n = 1, 2, \ldots. \tag{2.23}$$

This can be used to compute E_n recursively in terms of L_n and their derivatives.

Remark. Let Θ be an automorphism of $\mathfrak{R}$ which commutes with the partial derivative operators with respect to the variables t_n, and let us consider an extension of the linear system (2.12) of the form

$$\Theta(W) = KW. \tag{2.24}$$

This gives rise to the additional integrability conditions

$$\partial_{t_n}(K) = \Theta(L_n)K - KL_n, \qquad n = 1, 2, \ldots, \tag{2.25}$$

which can also be expressed as

$$K_{-[\lambda]}E(\lambda) = \Theta(E(\lambda))K. \tag{2.26}$$

If the elements of $\mathfrak{R}$ depend on a discrete variable, the shift operator Λ with respect to this variable provides us with an example of such a Θ. Then (2.24) is a discrete evolution equation.

2.3. Lax equations

Let

$$L := W\tilde{D}W^{-1} \tag{2.27}$$

where W satisfies the linear system (2.12) and $\tilde{D} \in \mathfrak{R}$ is independent of $\mathbf{t}$. This is known as a (Wilson–Sato) 'dressing transformation'. Differentiation of (2.27) with respect to t_n and use of (2.12) leads to the Lax equations

$$\partial_{t_n}(L) = [L_n, L], \qquad n = 1, 2, \ldots. \tag{2.28}$$

Typically, we should look for a recipe which determines L_n, E_n in terms of L (cf section 2.4). An alternative form of equations (2.28) is obtained as follows:

$$E(\lambda)LW = E(\lambda)W\tilde{D} = W_{-[\lambda]}\tilde{D} = (W\tilde{D})_{-[\lambda]} = (LW)_{-[\lambda]} = L_{-[\lambda]}W_{-[\lambda]} = L_{-[\lambda]}E(\lambda)W. \tag{2.29}$$

Hence,

$$L_{-[\lambda]}E(\lambda) = E(\lambda)L. \tag{2.30}$$

Expanding in λ, this becomes

$$\hat{\chi}_n(L) = [E_n, L] - \sum_{k=1}^{n-1} \hat{\chi}_k(L)E_{n-k}, \qquad n = 1, 2, \ldots. \tag{2.31}$$

Clearly, this set of equations is equivalent to (2.28).

2.4. Gelfand–Dickey-type hierarchies

Let $\mathfrak{R}$ now be a unital associative algebra with a projection $(\,)_-$ such that $\mathfrak{R} = (\mathfrak{R})_- \oplus (\mathfrak{R})_+$ where $(\,)_+ = \mathrm{id} - (\,)_-$ and $(\mathfrak{R})_-$, $(\mathfrak{R})_+$ are subalgebras. Furthermore, we assume that $\mathfrak{R}$ is generated by L via the product in $\mathfrak{R}$ and the projection $(\,)_-$.

If L_n is of the form

$$L_n = (L^n)_+, \tag{2.32}$$

we call (2.28) a *Gelfand–Dickey-type hierarchy*. In this case, a well-known argument (see [1], for example) shows that the zero curvature conditions (2.13) are satisfied as a consequence of (2.28), so that no further equations have to be added to those already given by (2.28).

In the following, we derive a very simple formula for E_n. Let us introduce $\tilde{E}_0 = 1$, $\tilde{E}_1 = -L$ and

$$\tilde{E}_{n+1} = (\tilde{E}_n)_- L, \qquad n = 1, 2, \ldots. \tag{2.33}$$

Lemma 2.1.

$$\tilde{E}_n = -\sum_{k=1}^{n} (\tilde{E}_{n-k})_+ L^k \qquad n = 1, 2, \ldots. \tag{2.34}$$

Proof. Using

$$(\tilde{E}_{n-k})_+ L^k = \tilde{E}_{n-k} L^k - (\tilde{E}_{n-k})_- L^k = \tilde{E}_{n-k} L^k - \tilde{E}_{n-k+1} L^{k-1}$$

for $k = 1, \ldots, n-1$, we have the following telescoping sum:

$$\sum_{k=1}^{n-1} (\tilde{E}_{n-k})_+ L^k = \sum_{k=1}^{n-1} \tilde{E}_{n-k} L^k - \sum_{k=0}^{n-2} \tilde{E}_{n-k} L^k = -\tilde{E}_n + \tilde{E}_1 L^{n-1} = -\tilde{E}_n - L^n.$$

□

It is convenient to introduce the product (see also [25, 26], for example)

$$X \vartriangle Y := (X)_+ Y - X (Y)_- = (X)_+ (Y)_+ - (X)_- (Y)_- \tag{2.35}$$

for $X, Y \in \mathfrak{R}$.

Lemma 2.2. *As a consequence of the hierarchy (2.28) with (2.32), we have*

$$n\tilde{E}_n = -L^n - \sum_{k=1}^{n-1} \big(\partial_{t_k}(\tilde{E}_{n-k}) + \tilde{E}_{n-k} \vartriangle L^k \big), \qquad n = 1, 2, \ldots. \tag{2.36}$$

Proof. By induction on n. The formula trivially holds for $n = 1$ and is easily verified for $n = 2$ using $L_{t_1} = [(L)_+, L]$. Let us assume that it holds for n. Then

$$n\tilde{E}_{n+1} = n(\tilde{E}_n)_- L = -(L^n)_- L - \sum_{k=1}^{n-1} \big(\partial_{t_k}(\tilde{E}_{n-k})_- L - (\tilde{E}_{n-k})_- (L^k)_- L \big)$$

by the use of the induction hypothesis. With the help of

$$\begin{aligned}\partial_{t_k}(\tilde{E}_{n-k})_- L &= \partial_{t_k}(\tilde{E}_{n+1-k}) - (\tilde{E}_{n-k})_- \partial_{t_k}(L) = \partial_{t_k}(\tilde{E}_{n+1-k}) + (\tilde{E}_{n-k})_- [(L^k)_-, L] \\ &= \partial_{t_k}(\tilde{E}_{n+1-k}) + (\tilde{E}_{n-k})_- (L^k)_- L - \tilde{E}_{n+1-k} (L^k)_-,\end{aligned}$$

we obtain

$$n\tilde{E}_{n+1} = -(L^n)_- L - \sum_{k=1}^{n-1} \big(\partial_{t_k}(\tilde{E}_{n+1-k}) - \tilde{E}_{n+1-k}(L^k)_-\big)$$
$$= -\sum_{k=1}^{n} \big(\partial_{t_k}(\tilde{E}_{n+1-k}) - \tilde{E}_{n+1-k}(L^k)_-\big)$$

where we again made use of (2.28). Finally, we take account of

$$\sum_{k=1}^{n} \tilde{E}_{n+1-k}(L^k)_- = -\sum_{k=1}^{n} \tilde{E}_{n+1-k} \vartriangle L^k + \sum_{k=1}^{n} (\tilde{E}_{n+1-k})_+ L^k$$
$$= -\sum_{k=1}^{n} \tilde{E}_{n+1-k} \vartriangle L^k + \sum_{k=1}^{n+1} (\tilde{E}_{n+1-k})_+ L^k - (\tilde{E}_0)_+ L^{n+1}$$
$$= -\sum_{k=1}^{n} \tilde{E}_{n+1-k} \vartriangle L^k - \tilde{E}_{n+1} - L^{n+1},$$

where we applied (2.34) in the last step, to obtain (2.36) for $n + 1$. □

Theorem 2.1. *For a Gelfand–Dickey-type hierarchy, we have*

$$E_n = (\tilde{E}_n)_+, \qquad n = 0, 1, 2, \ldots. \tag{2.37}$$

Proof. This is clearly true for $n = 0, 1$. Taking the $(\,)_+$ part of (2.36) leads to

$$n(\tilde{E}_n)_+ = -\sum_{k=1}^{n} \big(\partial_{t_k}(\tilde{E}_{n-k})_+ + (\tilde{E}_{n-k})_+(L^k)_+\big).$$

Now our assertion follows by comparison with the recursion relation (2.23) for E_n. □

3. A class of examples

In this section, we specialize the very general setting of the previous section in order to make contact with some known hierarchies. Our basic assumptions are formulated in the first subsection below. An important tool is the notion of residue (see [1], for example) exploited in section 3.2. With its help we derive a general 'functional representation' in section 3.3, which also presents several examples.

3.1. Preliminaries

Let $\mathcal{A}$ be a unital associative algebra and D an invertible linear operator on $\mathcal{A}$ such that

(1) all its powers D^n, $n \in \mathbb{Z}$, are linearly independent (in the sense of a left $\mathcal{A}$-module),
(2) for all $a \in \mathcal{A}$,

$$Da = \Theta(a)D + \vartheta(a). \tag{3.1}$$

Then $\Theta : \mathcal{A} \to \mathcal{A}$ has to be an algebra endomorphism and ϑ a Θ-twisted derivation,

$$\vartheta(ab) = \vartheta(a)b + \Theta(a)\vartheta(b). \tag{3.2}$$

(3) D and Θ are invertible (hence Θ is an automorphism of $\mathcal{A}$).
(4) D commutes with all partial derivatives with respect to a set of coordinates, say t_n, $n \in \mathbb{N}$. This implies that also Θ and ϑ commute with all these partial derivatives.

As a consequence of conditions (2) and (3), we have

$$aD^{-1} = D^{-1}\Theta(a) + D^{-1}\vartheta(a)D^{-1} \tag{3.3}$$

and thus

$$\begin{aligned} D^{-1}a &= \Theta^{-1}(a)D^{-1} - D^{-1}\vartheta(\Theta^{-1}(a))D^{-1} \\ &= \Theta^{-1}(a)D^{-1} - \Theta^{-1}(\vartheta(\Theta^{-1}(a)))D^{-2} + D^{-1}(\vartheta\circ\Theta^{-1})^2(a)D^{-2}. \end{aligned} \tag{3.4}$$

Iteration leads to

$$\begin{aligned} D^{-1}a &= \Theta^{-1}(a)D^{-1} - \vartheta_1(a)D^{-2} + \vartheta_2(a)D^{-3} - \cdots \\ &= \Theta^{-1}(a)D^{-1} + \sum_{1\leqslant n}(-1)^n\vartheta_n(a)D^{-n-1}, \qquad \forall\, a\in\mathcal{A} \end{aligned} \tag{3.5}$$

where

$$\vartheta_n := \Theta^{-1}\circ(\vartheta\circ\Theta^{-1})^n, \qquad n = 1, 2, \ldots. \tag{3.6}$$

Examples. Let $\mathcal{A}$ be an algebra of matrices of functions.

(1) Let D be the operator of multiplication by a parameter ζ. The integer powers of ζ are linearly independent and commute with all $a\in\mathcal{A}$. We have $\Theta = \mathrm{id}$ and $\vartheta = 0$.
(2) $D = \partial$, the operator of partial differentiation with respect to a variable x. Then $\partial a = a_x + a\partial$ and $\partial^{-1}a = a\partial^{-1} - a_x\partial^{-2} + a_{xx}\partial^{-3} - \cdots$, where an index x indicates a partial derivative with respect to x (see [1], for example). Here, we have $\Theta = \mathrm{id}$ and $\vartheta = \partial_x$.
(3) $D = \Lambda$, the shift operator $(\Lambda a)(x) = a(x+1)$ acting on a function a (or a matrix of functions) of a variable x. Then $\Lambda a = a^+\Lambda$ and $\Lambda^{-1}a = a^-\Lambda^{-1}$ where $a^{\pm}(x) = a(x\pm 1)$. In this case, $\Theta = \Lambda$ and $\vartheta = 0$.
(4) $D = \Lambda_q$, where $q\notin\{0, 1\}$ and $(\Lambda_q a)(x) = a(qx)$ acting on a function a (or a matrix of functions) of a variable x. Here we have $\Theta = \Lambda_q$ and $\vartheta = 0$.
(5) Let D be the q-derivative operator

$$(\partial_q a)(x) = \frac{a(qx) - a(x)}{x(q-1)} \tag{3.7}$$

acting on functions of a variable x. In this case ϑ is the q-derivative and $\Theta = \Lambda_q$ with the q-shift operator defined above.

Let $u_k\in\mathcal{A}$ and $\mathfrak{R}$ be the algebra generated by the formal series

$$L = Wu_0DW^{-1} = u_0D + u_1 + u_2D^{-1} + u_3D^{-2} + \cdots \tag{3.8}$$

(where we assume $u_0\neq 0$) and the projections

$$(X)_- = X_{<0}, \qquad (X)_+ = X_{\geqslant 0} \tag{3.9}$$

of an element $X\in\mathfrak{R}$ to its parts containing only negative, respectively non-negative, powers of D. Another choice would be $(X)_- = X_{<1}$, $(X)_+ = X_{\geqslant 1}$. This can be treated analogously and leads to further examples, see also section 4. As a consequence of our assumptions for the operator D, we have $\mathfrak{R}_+\mathfrak{R}_+\subset\mathfrak{R}_+$ and $\mathfrak{R}_-\mathfrak{R}_-\subset\mathfrak{R}_-$, as required in section 2.4. In the following, we consider Gelfand–Dickey-type hierarchies in this specialized framework.

Remark. Generically, the set of zero curvature equations (2.13), with (2.32) and (3.8), actually implies the Lax hierarchy (2.28) and is then *equivalent* to it. The following argument is taken from [27]. Writing (2.13) in the form

$$\partial_{t_n}(L^m) - [(L^n)_+, L^m] = \partial_{t_n}(L^m)_- + \partial_{t_m}(L^n)_+ - [(L^n)_+, (L^m)_-], \tag{3.10}$$

we observe that, for fixed n, on the right-hand side the order of powers of D is bounded above by n, whereas on the left-hand side it increases with m. Suppose $X^{(n)} := \partial_{t_n}(L) - [(L^n)_+, L] \neq 0$. The left-hand side of (3.10) then takes the form

$$\partial_{t_n}(L^m) - [(L^n)_+, L^m] = \sum_{k=0}^{m-1} L^k X^{(n)} L^{m-1-k}, \qquad m \geqslant 1,$$

which, for sufficiently large m, contains terms with powers of D greater than n, which leads to a contradiction (unless the coefficients of all those terms vanish because of very special properties of L). Hence, $X^{(n)} = 0$ and thus $\partial_{t_n}(L) - [(L^n)_+, L] = 0$.

As a consequence of (2.28) with (2.32), we have

$$\partial_{t_n}(u_0) = 0, \qquad n = 1, 2, \ldots \tag{3.11}$$

so that u_0 has to be constant. Furthermore, the first hierarchy equation in particular leads to

$$u_{1,x} = u_0 \Theta(u_2) - u_2 \Theta^{-1}(u_0). \tag{3.12}$$

In the following, we will look at u_2 as our 'primary object'. Introducing a potential ϕ such that

$$u_2 = \phi_x, \tag{3.13}$$

equation (3.12) becomes

$$u_1 = u_0 \Theta(\phi) - \phi \Theta^{-1}(u_0) \tag{3.14}$$

(up to addition of an arbitrary element of $\mathcal{A}$ independent of x, which we set to zero).

In the following, we use the abbreviations

$$a^+ := \Theta(a), \qquad a^- := \Theta^{-1}(a) \tag{3.15}$$

where $a \in \mathcal{A}$.

3.2. Taking residues

We define the residue $\mathrm{res}(X)$ of $X \in \mathfrak{R}$ as the *left* coefficient[6] of D^{-1}. It follows that

$$(LX_{<0})_{\geqslant 0} = u_0 \Theta(\mathrm{res}(X)), \qquad (X_{<0}L)_{\geqslant 0} = \mathrm{res}(X)\Theta^{-1}(u_0). \tag{3.16}$$

The zero curvature condition (2.13) with (2.32) can be written as follows:

$$\partial_{t_n}(L^m)_{<0} - \partial_{t_m}(L^n)_{<0} = [(L^m)_{<0}, (L^n)_{<0}]. \tag{3.17}$$

Taking the residue leads to

$$\mathrm{res}(L^m)_{t_n} = \mathrm{res}(L^n)_{t_m}. \tag{3.18}$$

Hence there is a $\phi \in \mathcal{A}$ such that

$$\phi_{t_n} = \mathrm{res}(L^n). \tag{3.19}$$

For $n = 1$, this is (3.13).

Lemma 3.1. *As a consequence of the hierarchy (2.28) with (2.32), we have*

$$\mathrm{res}(\tilde{E}_n) = \hat{\chi}_n(\phi), \qquad n = 1, 2, \ldots. \tag{3.20}$$

[6] This means that, before reading off the coefficient, we have to commute all powers of D to the right. If D is given by example 1 or 2 in subsection 3.1, the residue does not depend on the ordering, however.

Proof. First we note that

$$\mathrm{res}(X \vartriangle Y) = \mathrm{res}(X_{\geqslant 0} Y_{\geqslant 0}) - \mathrm{res}(X_{<0} Y_{<0})$$

vanishes for all $X, Y \in \mathfrak{R}$, since the first residue on the right-hand side vanishes as a consequence of $\mathfrak{R}_{\geqslant 0}\mathfrak{R}_{\geqslant 0} \subset \mathfrak{R}_{\geqslant 0}$ and the second vanishes because $X_{<0}Y_{<0}$ does not contain higher than -2 powers of D according to our assumptions for D. Also taking (3.19) into account, the residue of (2.36) is

$$n\,\mathrm{res}(\tilde{E}_n) = -\phi_{t_n} - \sum_{k=1}^{n-1} \partial_{t_k}(\mathrm{res}(\tilde{E}_{n-k})).$$

Our assertion now follows by comparing this recursion formula with (2.9), since $\mathrm{res}(\tilde{E}_1) = -\mathrm{res}(L) = \hat{\chi}_1(\phi)$. □

Lemma 3.2.

$$E(\lambda) = 1 - \lambda u_0 D - \lambda(u_0\phi^+ - \phi_{-[\lambda]}u_0^-). \tag{3.21}$$

Proof. As a consequence of theorem 2.1, equation (2.33) and lemma 3.1, we have

$$E_{n+1} = ((\tilde{E}_n)_{<0}L)_{\geqslant 0} = ((\tilde{E}_n)_{<0}u_0D)_{\geqslant 0} = (\mathrm{res}(\tilde{E}_n)D^{-1}u_0D)_{\geqslant 0} = \mathrm{res}(\tilde{E}_n)u_0^- = \hat{\chi}_n(\phi)u_0^-$$

for $n = 1, 2, \ldots$. Hence,

$$E(\lambda) = 1 - \lambda u_0 D - \lambda(\phi - \phi_{-[\lambda]})u_0^- - \lambda u_1$$

from which our assertion follows by the use of (3.14). □

Remark. Note that (3.21) is *polynomial* in D. If we express the linear system (2.12) in the form $\chi_n(W) = H_n W$ with $H_n \in \mathfrak{R}$, instead of (2.14), the resulting relation $E(\lambda)_{[\lambda]}H(\lambda) = 1$ with $H(\lambda) = \sum_{n\geqslant 0}\lambda^n H_n$ implies that $H(\lambda)$ is an *infinite* formal power series in D. This is the reason why we chose to work with $E(\lambda)$ instead of $H(\lambda)$.

3.3. Functional representations

The next result evaluates equations (2.18) in the framework under consideration. Since by construction they are equivalent to the zero curvature equations (2.13), according to the remark in section 3.1 they are generically also equivalent to the complete hierarchy.

Theorem 3.1. *In the present context, (2.18) is equivalent to*

$$\begin{aligned} u_0\vartheta\big((\phi_{[\lambda_1]} - \phi_{[\lambda_2]})u_0^-\big) &= \left(\frac{1}{\lambda_1} - u_0\phi^+_{[\lambda_1]} + \phi u_0^-\right)\left(\frac{1}{\lambda_2} - u_0\phi^+_{[\lambda_1]+[\lambda_2]} + \phi_{[\lambda_1]}u_0^-\right) \\ &\quad - \left(\frac{1}{\lambda_2} - u_0\phi^+_{[\lambda_2]} + \phi u_0^-\right)\left(\frac{1}{\lambda_1} - u_0\phi^+_{[\lambda_1]+[\lambda_2]} + \phi_{[\lambda_2]}u_0^-\right) \end{aligned} \tag{3.22}$$

(which is antisymmetric in λ_1, λ_2).

Proof. Let us write (3.21) as

$$-\frac{1}{\lambda}E(\lambda) = u_0 D - \omega(\lambda) \qquad \text{where} \quad \omega(\lambda) := \frac{1}{\lambda} - u_0\phi^+ + \phi_{-[\lambda]}u_0^-.$$

Using this in (2.18), we obtain the following two equations:

$$u_0(\omega^+(\lambda_1) - \omega^+(\lambda_2)) = (\omega(\lambda_1)_{-[\lambda_2]} - \omega(\lambda_2)_{-[\lambda_1]})u_0,$$
$$u_0\vartheta(\omega(\lambda_2) - \omega(\lambda_1)) = \omega(\lambda_1)_{-[\lambda_2]}\omega(\lambda_2) - \omega(\lambda_2)_{-[\lambda_1]}\omega(\lambda_1).$$

The first equation turns out to be an identity by the use of the definition of $\omega(\lambda)$. So, we are left with the second equation which is

$$u_0\vartheta\left(\left(\phi_{-[\lambda_2]}-\phi_{-[\lambda_1]}\right)u_0^-\right)=\left(\frac{1}{\lambda_1}-u_0\phi^+_{-[\lambda_2]}+\phi_{-[\lambda_1]-[\lambda_2]}u_0^-\right)\left(\frac{1}{\lambda_2}-u_0\phi^++\phi_{-[\lambda_2]}u_0^-\right)$$
$$-\left(\frac{1}{\lambda_2}-u_0\phi^+_{-[\lambda_1]}+\phi_{-[\lambda_1]-[\lambda_2]}u_0^-\right)\left(\frac{1}{\lambda_1}-u_0\phi^++\phi_{-[\lambda_1]}u_0^-\right).$$

After a Miwa shift $\mathbf{t}\to\mathbf{t}+[\lambda_1]+[\lambda_2]$, this becomes (3.22). □

To order $\lambda_2^0\lambda_1^n$, (3.22) yields

$$\chi_{n+1}(u_1)-u_0\chi_n\left(\phi^+_{t_1}-\vartheta\left(\phi u_0^-\right)\right)=u_1\chi_n(u_1)+\sum_{k=1}^{n-1}u_0\chi_k(\phi^+)\chi_{n-k}(u_1)+[u_0\chi_n(\phi^+),u_1],\tag{3.23}$$

and to order $\lambda_2^m\lambda_1^n$, $m,n\geqslant 1$,

$$\chi_{n+1}(\chi_m(\varphi))-\chi_{m+1}(\chi_n(\varphi))=\sum_{k=1}^{n}\chi_k(\varphi)\chi_{n-k}(\chi_m(\varphi))-\sum_{k=1}^{m}\chi_k(\varphi)\chi_{m-k}(\chi_n(\varphi))\tag{3.24}$$

where we introduced

$$\varphi:=\Theta^{-1}(u_0)\phi=u_0^-\phi.\tag{3.25}$$

In particular, for $m=1,n=2$, we recover the potential KP equation

$$\tfrac{1}{3}\varphi_{tx}-\tfrac{1}{12}\varphi_{xxxx}-\tfrac{1}{4}\varphi_{yy}=\tfrac{1}{2}(\varphi_x\varphi_x)_x-\tfrac{1}{2}[\varphi_x,\varphi_y]\tag{3.26}$$

where $x=t_1,y=t_2$ and $t=t_3$. In fact, as expressed in the subsequent theorem, the equations (3.24) are actually equivalent to the whole (noncommutative) potential KP hierarchy with the dependent variable (3.25).

Theorem 3.2. *The equations (3.22) imply*

$$\left(\lambda_1^{-1}-\lambda_2^{-1}+\varphi_{[\lambda_2]}-\varphi_{[\lambda_1]}\right)_x=\left(\lambda_1^{-1}-\lambda_2^{-1}+\varphi_{[\lambda_2]}-\varphi_{[\lambda_1]}\right)\left(\varphi_{[\lambda_1]+[\lambda_2]}-\varphi_{[\lambda_1]}-\varphi_{[\lambda_2]}+\varphi\right)$$
$$-\left[\varphi_{[\lambda_1]}-\varphi,\varphi_{[\lambda_2]}-\varphi\right],\tag{3.27}$$

which is equivalent to

$$\sum_{i,j,k=1}^{3}\epsilon_{ijk}\left(\lambda_i^{-1}(\varphi_{[\lambda_i]}-\varphi)+\varphi\varphi_{[\lambda_i]}\right)_{[\lambda_k]}=0,\tag{3.28}$$

where $\lambda_1,\lambda_2,\lambda_3$ are independent parameters and ϵ_{ijk} is totally antisymmetric with $\epsilon_{123}=1$.

Proof. By expansion of (3.27) in λ_1,λ_2, one recovers (3.24), which we derived from (3.22). Summing (3.27) three times with cyclically permuted parameters $\lambda_1,\lambda_2,\lambda_3$ leads to

$$\lambda_1^{-1}\left(\left(\varphi_{[\lambda_1]}-\varphi\right)_{[\lambda_3]}-\left(\varphi_{[\lambda_1]}-\varphi\right)_{[\lambda_2]}\right)+\lambda_2^{-1}\left(\left(\varphi_{[\lambda_2]}-\varphi\right)_{[\lambda_1]}-\left(\varphi_{[\lambda_2]}-\varphi\right)_{[\lambda_3]}\right)$$
$$+\lambda_3^{-1}\left(\left(\varphi_{[\lambda_3]}-\varphi\right)_{[\lambda_2]}-\left(\varphi_{[\lambda_3]}-\varphi\right)_{[\lambda_1]}\right)$$
$$=\left[\varphi\left(\varphi_{[\lambda_1]}-\varphi_{[\lambda_3]}\right)\right]_{[\lambda_2]}+\left[\varphi\left(\varphi_{[\lambda_2]}-\varphi_{[\lambda_1]}\right)\right]_{[\lambda_3]}+\left[\varphi\left(\varphi_{[\lambda_3]}-\varphi_{[\lambda_2]}\right)\right]_{[\lambda_1]}$$

which can be rearranged to (3.28). The limit $\lambda_3\to 0$ leads back to (3.27). □

As shown in [20], (3.27) is a 'noncommutative' version of the *differential Fay identity* for the (potential) KP hierarchy (see [3–9], for example).[7] Equation (3.28), which already

[7] In the commutative case, setting $\varphi=\tau_x/\tau$ and integrating once leads to the familiar differential Fay identity. The form in which we wrote (3.27) facilitates this calculation.

appeared in [11, 12], is then a 'noncommutative' version of the *algebraic Fay identity*. Here we have shown that, expressed as above in terms of (3.25), these formulae apply universally to *all* examples in the class considered in this section!

In the special case of the KP hierarchy, (3.27) is actually equivalent to the hierarchy equations. This is *not* true in general. Typically (3.22) contains equations beyond those given by (3.27) and these are given by (3.23).

Remark. The KP hierarchy equations in the form (3.24) are the integrability conditions of the linear system

$$\chi_{m+1}(f) = -f\,\chi_m(\varphi), \qquad m = 1, 2, \ldots. \tag{3.29}$$

In fact, as a consequence of the latter we have (by the use of (2.11))

$$\chi_{n+1}(\chi_{m+1}(f)) = f\left(-\chi_{n+1}(\chi_m(\varphi)) + \sum_{k=1}^{n} \chi_k(\varphi)\chi_{n-k}(\chi_m(\varphi))\right) - \chi_1(f)\chi_n(\chi_m(\varphi)). \tag{3.30}$$

Antisymmetrization in m, n yields (3.24).

3.3.1. KP hierarchy. The usual KP hierarchy in the Gelfand–Dickey framework (see [1], for example) is obtained by choosing $u_0 = 1, u_1 = 0$, $D = \partial$, the operator of partial differentiation with respect to $x = t_1$, so that $\Theta = \mathrm{id}$ and $\vartheta = \partial_x$. Then (3.22) becomes

$$\begin{aligned} -\big(\phi_{[\lambda_1]} - \phi_{[\lambda_2]}\big)_x &= \big(\lambda_2^{-1} - \phi_{[\lambda_2]} + \phi\big)\big(\lambda_1^{-1} - \phi_{[\lambda_1]+[\lambda_2]} + \phi_{[\lambda_2]}\big) \\ &\quad - \big(\lambda_1^{-1} - \phi_{[\lambda_1]} + \phi\big)\big(\lambda_2^{-1} - \phi_{[\lambda_1]+[\lambda_2]} + \phi_{[\lambda_1]}\big) \end{aligned} \tag{3.31}$$

which, after some simple algebraic manipulations, yields (3.27) (note that $\varphi = \phi$ in this case). Hence, here (3.22) (and thus (2.18)) reduces to (3.27).

3.3.2. Discrete KP and q-KP. The choice of D in example 3 leads to the discrete KP hierarchy [6, 28, 29]. For the choices of D in examples 4 and 5, the hierarchy has been called 'Frenkel system' [30] and 'KLR system' [31], respectively, in [6], where the authors proved that both are isomorphic to the discrete KP hierarchy (see also [1]).[8]

In the following we concentrate on examples 3 and 4, which can be treated simultaneously. Then $\vartheta = 0$ and $a^+ = \Theta(a) = \Lambda a \Lambda^{-1}$ with $(\Lambda a)(s) = a(s+1)$ or $(\Lambda a)(s) = a(qs)$. Furthermore, we choose $u_0 = 1$. Then (3.22) takes the form

$$\big(\lambda_2^{-1} - \phi^+_{[\lambda_2]} + \phi\big)\big(\lambda_1^{-1} - \phi^+_{[\lambda_1]+[\lambda_2]} + \phi_{[\lambda_2]}\big) = \big(\lambda_1^{-1} - \phi^+_{[\lambda_1]} + \phi\big)\big(\lambda_2^{-1} - \phi^+_{[\lambda_1]+[\lambda_2]} + \phi_{[\lambda_1]}\big). \tag{3.32}$$

In the limit $\lambda_2 \to 0$, this yields

$$\big(\lambda^{-1} - \phi^+_{[\lambda]} + \phi\big)_x = (\phi^+ - \phi)\big(\lambda^{-1} - \phi^+_{[\lambda]} + \phi\big) - \big(\lambda^{-1} - \phi^+_{[\lambda]} + \phi\big)\big(\phi^+_{[\lambda]} - \phi_{[\lambda]}\big). \tag{3.33}$$

Since according to theorem 3.2, ϕ and ϕ^+ both have to satisfy the KP hierarchy equations, the last equation should represent a *Bäcklund transformation* of the KP hierarchy. Let us momentarily turn to the case of a commutative algebra $\mathcal{A}$. Setting $\phi = \tau_x/\tau$ with a function τ, an integration leads to

$$\tau^+_{[\lambda],x}\tau - \tau^+_{[\lambda]}\tau_x = \lambda^{-1}\tau^+_{[\lambda]}\tau + \beta\tau^+\tau_{[\lambda]} \tag{3.34}$$

where β is an arbitrary x-independent function. This equation has a limit as $\lambda \to 0$ if

$$\beta = -\lambda^{-1} + \beta_0 + \beta_1\lambda + \cdots \tag{3.35}$$

[8] The authors of [31] actually amputated the q-derivative by dropping the argument x in the denominator of (3.7). See also [32–36] for work on q-deformed KP hierarchies.

where $\beta_0, \beta_1, \ldots$ are arbitrary x-independent functions. If the latter are all set to zero, (3.34) becomes equation (0.20) in [6] after a Miwa shift $\mathbf{t} \mapsto \mathbf{t} - [\lambda]$. Treating them as parameters, however, we should recover from (3.34) auto-Bäcklund transformations of the KP hierarchy.

Returning to the 'noncommutative' case, expansion of (3.33) in powers of λ leads to

$$
\begin{aligned}
(\chi_{n+1} - \chi_n \chi_1)(\phi^+) - \chi_{n+1}(\phi) &= (\phi^+ - \phi)\chi_n(\phi^+ - \phi) \\
&\quad + \sum_{k=1}^{n-1} \chi_k(\phi^+)\chi_{n-k}(\phi^+ - \phi) + [\chi_n(\phi^+), \phi^+ - \phi].
\end{aligned} \tag{3.36}
$$

For $n = 1$, this is

$$
(\phi^+ - \phi)_y - (\phi^+ + \phi)_{xx} = 2\phi^+_x(\phi^+ - \phi) - 2(\phi^+ - \phi)\phi_x. \tag{3.37}
$$

In the 'commutative' case with $\phi = \tau_x/\tau$, this becomes

$$
\left(D_x^2 - D_y\right)\tau^+ \cdot \tau = \beta_0 \tau^+ \tau \tag{3.38}
$$

in terms of Hirota derivatives D_x, D_y. This equation is a well-known auto-Bäcklund transformation of the KP equation [37–39].

3.3.3. AKNS. With the choices of example 1, (3.22) reads

$$
\begin{aligned}
&\left(\lambda_2^{-1} - u_0\phi_{[\lambda_2]} + \phi u_0\right)\left(\lambda_1^{-1} - u_0\phi_{[\lambda_1]+[\lambda_2]} + \phi_{[\lambda_2]}u_0\right) \\
&\quad = \left(\lambda_1^{-1} - u_0\phi_{[\lambda_1]} + \phi u_0\right)\left(\lambda_2^{-1} - u_0\phi_{[\lambda_1]+[\lambda_2]} + \phi_{[\lambda_1]}u_0\right).
\end{aligned} \tag{3.39}
$$

Choosing moreover

$$
u_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \phi = \begin{pmatrix} p & q \\ -r & p' \end{pmatrix}, \tag{3.40}
$$

we obtain the following system:

$$
\begin{aligned}
0 = {} & \left(\lambda_1^{-1} - \lambda_2^{-1} + p_{[\lambda_2]} - p_{[\lambda_1]}\right)\left(p_{[\lambda_1]+[\lambda_2]} - p_{[\lambda_1]} - p_{[\lambda_2]} + p\right) \\
& + (qr)_{[\lambda_2]} - (qr)_{[\lambda_1]} - \left[p_{[\lambda_1]} - p, p_{[\lambda_2]} - p\right],
\end{aligned} \tag{3.41}
$$

$$
0 = \left(\lambda_1^{-1} - \lambda_2^{-1} + p_{[\lambda_2]} - p_{[\lambda_1]}\right)q_{[\lambda_1]+[\lambda_2]} + \lambda_2^{-1}q_{[\lambda_1]} - \lambda_1^{-1}q_{[\lambda_2]}, \tag{3.42}
$$

$$
0 = \lambda_1^{-1}\left(r_{[\lambda_1]} - r\right) - \lambda_2^{-1}\left(r_{[\lambda_2]} - r\right) + r\left(p_{[\lambda_1]} - p_{[\lambda_2]}\right), \tag{3.43}
$$

which leaves p' undetermined. In the limit $\lambda_2 \to 0$, this system yields

$$
(p_{[\lambda]} - p)_x = qr - (qr)_{[\lambda]}, \tag{3.44}
$$

$$
q_{[\lambda]} - q = \lambda q_{[\lambda],x} + \lambda(p_{[\lambda]} - p)q_{[\lambda]}, \tag{3.45}
$$

$$
r_{[\lambda]} - r = \lambda r_x - \lambda r(p_{[\lambda]} - p). \tag{3.46}
$$

Multiplying (3.45) by r from the right, (3.46) by $q_{[\lambda]}$ from the left, adding the resulting equations and using (3.44), we find

$$
(p_{[\lambda]} - p + \lambda q_{[\lambda]}r)_x = [p_{[\lambda]} - p + \lambda q_{[\lambda]}r, p_{[\lambda]} - p]. \tag{3.47}
$$

Expanding this equation in powers of λ, a simple induction argument shows that[9]

$$
p_{[\lambda]} - p = -\lambda q_{[\lambda]}r. \tag{3.48}
$$

[9] Constants of integration are set to zero.

Eliminating p from (3.45) and (3.46) with the help of this formula, we arrive at

$$q_{-[\lambda]} - q + \lambda q_x = \lambda^2 q r_{-[\lambda]} q, \qquad r_{[\lambda]} - r - \lambda r_x = \lambda^2 r q_{[\lambda]} r, \tag{3.49}$$

which is a 'functional representation' of the AKNS hierarchy [16, 17], generalized to the case where q and r are matrices with entries from any associative algebra. Expanding the above system in powers of λ leads in lowest order to

$$q_{t_2} = q_{xx} - 2qrq, \qquad r_{t_2} = -r_{xx} + 2rqr. \tag{3.50}$$

To next order in λ we obtain, after use of the first system,

$$q_{t_3} = q_{xxx} - 3(qrq_x + q_x rq), \qquad r_{t_3} = r_{xxx} - 3(rqr_x + r_x qr). \tag{3.51}$$

For example, we may choose q and r as $M \times N$ and $N \times M$ matrices, respectively, with entries from any, possibly noncommutative, associative algebra. In this way, (3.50) also covers the case of (coupled) vector nonlinear Schrödinger equations (if we replace t_2 by the imaginary variable $\imath t$).[10]

Remark. Equations (3.41), (3.42) and (3.43) should reduce completely to the system (3.49) of functional equations, since it contains the full set of hierarchy equations. We verify that this is indeed the case. First, we note that equations (3.49) imply

$$(qr)_{[\lambda]} - qr = \lambda (q_{[\lambda]} r)_x. \tag{3.52}$$

Introducing p such that $p_x = -qr$, this yields (3.48) after an x-integration. Inserting (3.48) into (3.43) turns it into

$$\lambda_1^{-1}\big(r_{[\lambda_1]} - r\big) + \lambda_1 r q_{[\lambda_1]} r = \lambda_2^{-1}\big(r_{[\lambda_2]} - r\big) + \lambda_2 r q_{[\lambda_2]} r, \tag{3.53}$$

which means that $\lambda^{-1}(r_{[\lambda]} - r) + \lambda r q_{[\lambda_1]} r$ is independent of λ. This is obviously equivalent to the second of equations (3.49). Inserting (3.48) into (3.42) transforms it into

$$\lambda_1^{-1}\big(q_{[\lambda_1]} - q\big)_{[\lambda_2]} - \lambda_2^{-1}\big(q_{[\lambda_2]} - q\big)_{[\lambda_1]} + \big(\lambda_1 q_{[\lambda_1]} - \lambda_2 q_{[\lambda_2]}\big) r q_{[\lambda_1]+[\lambda_2]} = 0, \tag{3.54}$$

which indeed holds as a consequence of the second of equations (3.49), and the integrability condition of (3.48), which is

$$\lambda_2^{-1}\big(\big(q_{[\lambda_1]} r\big)_{[\lambda_2]} - q_{[\lambda_1]} r\big) - \lambda_1^{-1}\big(\big(q_{[\lambda_2]} r\big)_{[\lambda_1]} - q_{[\lambda_2]} r\big) = 0. \tag{3.55}$$

A lengthier calculation shows that (3.41) also results from (3.49).

Remark. Our general results imply that

$$\varphi = u_0 \phi = \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}, \tag{3.56}$$

where $p_x = -qr$ according to (3.48), solves the KP hierarchy as a consequence of the AKNS hierarchy equations. Inspection of (3.27) then shows that p satisfies the KP hierarchy.

Remark. Since u_0 satisfies $u_0^2 = u_0$, the dressing relation $L = W u_0 \zeta W^{-1}$ implies $L^2 = W(u_0\zeta)^2 W^{-1} = \zeta L$. As a consequence, $L^n = \zeta^{n-1} L$, so that the Lax equations (2.28) take a more familiar form of (generalized) AKNS hierarchies [42–44].

[10] See also [40, 41] and references therein concerning nonlinear Schrödinger-type equations.

4. Modified KP hierarchy

In this section, we derive a functional representation of the *modified KP hierarchy* [45–48], which is given by

$$\partial_{t_n}(L) = [(L^n)_{\geqslant 1}, L], \qquad n = 1, 2, \ldots \tag{4.1}$$

where

$$L = \partial + u_1 + u_2\partial^{-1} + \cdots \tag{4.2}$$

with the operator ∂ of partial differentiation with respect to $x = t_1$ (cf example 2) and coefficients from some associative algebra $\mathcal{A}$. Obviously, $E_n, n > 0$, which are determined by theorem 2.1 and the recursion relation (2.33), are linearly homogeneous in ∂. Hence,

$$E(\lambda) = 1 - \lambda\omega(\lambda)\partial, \tag{4.3}$$

where

$$\omega(\lambda) = 1 + u_1\lambda + \big(u_2 + u_1^2\big)\lambda^2 + \cdots \tag{4.4}$$

is a power series in λ with coefficients in $\mathcal{A}$. Equation (2.18) now takes the form

$$\big(1 - \lambda_1\omega(\lambda_1)_{-[\lambda_2]}\partial\big)(1 - \lambda_2\omega(\lambda_2)\partial) = \big(1 - \lambda_2\omega(\lambda_2)_{-[\lambda_1]}\partial\big)(1 - \lambda_1\omega(\lambda_1)\partial). \tag{4.5}$$

Expansion in powers of ∂ leads to

$$\omega(\lambda_1)_{-[\lambda_2]}\omega(\lambda_2) = \omega(\lambda_2)_{-[\lambda_1]}\omega(\lambda_1) \tag{4.6}$$

and

$$\frac{1}{\lambda_2}\big(\omega(\lambda_1) - \omega(\lambda_1)_{-[\lambda_2]}\big) - \frac{1}{\lambda_1}\big(\omega(\lambda_2) - \omega(\lambda_2)_{-[\lambda_1]}\big)$$
$$= \omega(\lambda_2)_{-[\lambda_1]}\omega(\lambda_1)_x - \omega(\lambda_1)_{-[\lambda_2]}\omega(\lambda_2)_x. \tag{4.7}$$

The first equation is solved by

$$\omega(\lambda) = f_{-[\lambda]}f^{-1} \tag{4.8}$$

with some invertible element $f \in \mathcal{A}$. By comparison with (4.4),

$$v := u_1 = -f_x f^{-1}. \tag{4.9}$$

Next we use (4.8) in (4.7), multiply by $(f^{-1})_{-[\lambda_1]-[\lambda_2]}$ from the left and by f from the right, and apply a Miwa shift $\mathbf{t} \to \mathbf{t} + [\lambda_1] + [\lambda_2]$ to obtain the following functional representation of the mKP hierarchy:

$$\lambda_2^{-1}\big(\big(f^{-1}f_{[\lambda_2]}\big)_{[\lambda_1]} - f^{-1}f_{[\lambda_2]}\big) - \lambda_1^{-1}\big(\big(f^{-1}f_{[\lambda_1]}\big)_{[\lambda_2]} - f^{-1}f_{[\lambda_1]}\big)$$
$$= (f^{-1}f_x)_{[\lambda_1]} - (f^{-1}f_x)_{[\lambda_2]}, \tag{4.10}$$

which is a noncommutative version of the differential Fay identity of the mKP hierarchy. Adding it three times with cyclically permuted parameters $\lambda_1, \lambda_2, \lambda_3$ leads to the corresponding algebraic Fay identity

$$\lambda_1^{-1}\big(\big(f^{-1}f_{[\lambda_1]}\big)_{[\lambda_3]} - \big(f^{-1}f_{[\lambda_1]}\big)_{[\lambda_2]}\big) + \lambda_2^{-1}\big(\big(f^{-1}f_{[\lambda_2]}\big)_{[\lambda_1]} - \big(f^{-1}f_{[\lambda_2]}\big)_{[\lambda_3]}\big)$$
$$+ \lambda_3^{-1}\big(\big(f^{-1}f_{[\lambda_3]}\big)_{[\lambda_2]} - \big(f^{-1}f_{[\lambda_3]}\big)_{[\lambda_1]}\big) = 0. \tag{4.11}$$

(4.10) is equivalent to

$$\chi_n(f^{-1}\chi_{m+1}(f)) = \chi_m(f^{-1}\chi_{n+1}(f)), \qquad m, n = 1, 2, \ldots. \tag{4.12}$$

For $m = 1$ and $n = 2$, this yields

$$\left(f_t - \tfrac{1}{4} f_{xxx}\right)_x - \tfrac{3}{4} f_{yy} - f_x f^{-1}(f_t - f_{xxx}) + \tfrac{3}{4}\left(f_y f^{-1} + (f_x f^{-1})_x - (f_x f^{-1})^2\right)(f_{xx} + f_y) = 0 \tag{4.13}$$

which, multiplied by f^{-1} from the right, leads to

$$v_t - \tfrac{1}{4} v_{xxx} + \tfrac{3}{2} v v_x v - \tfrac{3}{4}[v, v_{xx}] - \tfrac{3}{4}\left(w_y + w(v_x - v^2) + (v_x + v^2)w\right) = 0 \tag{4.14}$$

where we introduced

$$w := -f_y f^{-1}. \tag{4.15}$$

Since $w_x = v_y - [v, w]$, there is no way to express also the terms involving w completely in terms of v, unless we assume that $\mathcal{A}$ is commutative, in which case we obtain the mKP equation (see [49], for example)

$$v_t - \tfrac{1}{4} v_{xxx} + \tfrac{3}{2} v^2 v_x - \tfrac{3}{4}\partial^{-1}(v_{yy}) - \tfrac{3}{2} v_x \partial^{-1}(v_y) = 0. \tag{4.16}$$

Returning to the noncommutative case and setting $f_y = 0$, (4.14) reduces to the second version of a 'matrix mKdV equation' in [50],

$$v_t - \tfrac{1}{4} v_{xxx} + \tfrac{3}{2} v v_x v - \tfrac{3}{4}[v, v_{xx}] = 0. \tag{4.17}$$

The mKP hierarchy in the form (4.12) implies the existence of a potential ϕ such that

$$f^{-1}\chi_{n+1}(f) = -\chi_n(\phi), \qquad n = 1, 2, \ldots, \tag{4.18}$$

which can, of course, also be written as

$$\chi_{n+1}(f) = -f\chi_n(\phi), \qquad n = 1, 2, \ldots. \tag{4.19}$$

Whereas the first form of this linear system naturally imposes integrability conditions on f, namely the mKP equations in the form (4.12), the second gives rise to integrability conditions in terms of the potential ϕ. Since (4.19) coincides with (3.29), the latter are precisely the potential KP hierarchy equations in the form (3.24) (with φ replaced by ϕ).

Furthermore, the above linear system mediates between the two hierarchies. The lowest ($n = 1$) member of (4.19) reads

$$u := \phi_x = -f^{-1}\chi_2(f) = -\tfrac{1}{2} f^{-1}(f_y + f_{xx}). \tag{4.20}$$

In the 'commutative' case, we can express the right-hand side in terms of v and recover the Miura transformation (see [12, 51], for example), which maps solutions of the mKP to solutions of the KP equation.

Remark. The 'duality' between the mKP and the KP hierarchy, which emerged here, reminds us of the relation between different forms of the (anti-) self-dual Yang–Mills equation (see [52], for example) and also the analogous relation between the principal chiral model and its pseudo-dual. In the latter case, the analogue of (4.19) is

$$\partial_{t_{n+1}}(f) = -f\partial_{t_n}(\phi), \qquad n = 1, 2, \ldots \tag{4.21}$$

which gives rise to the following two versions of integrability conditions:

$$\partial_{t_n}\left(f^{-1}\partial_{t_{m+1}}(f)\right) = \partial_{t_m}\left(f^{-1}\partial_{t_{n+1}}(f)\right) \tag{4.22}$$

(the analogue of the mKP hierarchy equations in the form (4.12)) and

$$\partial_{t_{n+1}}\partial_{t_m}(\phi) - \partial_{t_{m+1}}\partial_{t_n}(\phi) = \left[\partial_{t_n}(\phi), \partial_{t_m}(\phi)\right] \tag{4.23}$$

(the analogue of the KP hierarchy equations in the form obtained from (4.19)). The condition $t_{n+2} = t_n$ reduces these systems to the principal chiral model equation

$$\partial_{t_2}\left(f^{-1}\partial_{t_2}(f)\right) = \partial_{t_1}\left(f^{-1}\partial_{t_1}(f)\right), \tag{4.24}$$

respectively the pseudodual chiral model equation [53, 54]

$$\partial_{t_1}\partial_{t_1}(\phi) - \partial_{t_2}\partial_{t_2}(\phi) = [\partial_{t_2}(\phi), \partial_{t_1}(\phi)]. \tag{4.25}$$

Both are known to be expressions of the system

$$F := \mathrm{d}A + A \wedge A = 0, \qquad \mathrm{d} \star A = 0, \tag{4.26}$$

for a 1-form A in two dimensions, where $\star$ is the Euclidean Hodge operator. The chiral model results by solving the first equation by $A = f^{-1}\,\mathrm{d}f$ with an invertible matrix f. The pseudodual model is obtained by solving the second equation via $A = \star\mathrm{d}\phi$ with a matrix of functions ϕ. As we have seen, both models also emerge from the linear system (4.21). The latter has the following generalization:

$$\partial_{s_n}(f) = -f\,\partial_{t_n}(\phi) \tag{4.27}$$

with an additional set of independent variables s_n, $n = 1, 2, \ldots$. The corresponding integrability conditions are

$$\partial_{t_n}\big(f^{-1}\partial_{s_m}(f)\big) = \partial_{t_m}\big(f^{-1}\partial_{s_n}(f)\big), \tag{4.28}$$

respectively,

$$\partial_{s_n}\partial_{t_m}(\phi) - \partial_{s_m}\partial_{t_n}(\phi) = \big[\partial_{t_n}(\phi), \partial_{t_m}(\phi)\big]. \tag{4.29}$$

(4.28) represents a self-dual Yang–Mills hierarchy [52, 55–58] and (4.29) is its 'dual' version. It remains to be seen whether there is a meaningful 'reverse' analogue of (4.27) which generalizes (4.19) in a similar way as (4.27) generalizes (4.21).

5. Conclusions

In this work, we formulated a rather general approach towards 'functional representations' of integrable hierarchies, in particular analogues of 'Fay identities'. This formalism is *not* restricted to commutative dependent variables, but genuinely applies to 'noncommutative hierarchies', where the dependent variables live in any noncommutative algebra[11], like matrix KP hierarchies. The central part of the formalism is general enough to embrace many more integrable hierarchies and should serve to unify individual results in the literature. We provided corresponding examples, but by far did not exhaust the possibilities.

Apart from the fact that our approach presents a fairly simple and systematic way towards functional representations of specific 'noncommutative' integrable hierarchies, it also serves beyond that as a tool in 'integrable hierarchy theory'. For example, the application of the formalism in section 4 nicely displays the 'duality' between the mKP and the KP hierarchy.

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[11] This includes the case of an algebra of functions supplied with a Moyal-product, for example.

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